Nonlinear recurrence which equivalent to linear.

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Let $a_n \neq 0$ be a real sequence such that $a_{n+1} = \dfrac{a_n^2 - 1}{a_{n-1}}, n \in \mathbb{N}$.

Prove there exists a real number α such that $a_{n+1} = \alpha a_n - a_{n-1}, n \in \mathbb{N}$.

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Since $a_{n+1} = \dfrac{a_n^2 - 1}{a_{n-1}} \Leftrightarrow a_{n+1}a_{n-1} + 1 = a_n^2$ and $a_n \neq 0$ for any $n \in \mathbb{N}$ then

$a_{n+2}a_n + 1 - (a_{n+1}a_{n-1} + 1) = a_{n+1}^2 - a_n^2 \Leftrightarrow a_{n+2}a_n + a_n^2 = a_{n+1}a_{n-1} + a_{n+1}^2 \Leftrightarrow$

$\dfrac{a_{n+2} + a_n}{a_{n+1}} = \dfrac{a_{n+1} + a_{n-1}}{a_n}, \forall n \in \mathbb{N}$ and, therefore, $\dfrac{a_{n+1} + a_{n-1}}{a_n} = \dfrac{a_2 + a_0}{a_1} \Leftrightarrow$

$a_{n+1} = \alpha a_n - a_{n-1}, n \in \mathbb{N}$, where $\alpha = \dfrac{a_2 + a_0}{a_1} = \dfrac{\dfrac{a_1^2 - 1}{a_0} + a_0}{a_1} = \dfrac{a_0^2 + a_1^2 - 1}{a_1 a_0}$